Manipulability Comparisons of Positional Voting Rules for Large Three-Candidates Elections

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Abstract Given any positional voting rule (PVR), a characterization of unstable voting situations is provided for three-candidate elections with large electorates. Our contribution to this literature concerns three aspects: (i) a comparison of PVRs via set inclusion of their corresponding sets of unstable voting situations; (ii) a full classification of PVRs according to their vulnerability to coalitional manipulation; and (iii) the determination of the maximal minimum size of a manipulating coalition. The attractiveness of the results obtained is that our conclusions are valid for the whole class of PVRs. It turns out that, under the Impartial Anonymous Culture (IAC), the PVR that minimizes the frequency of unstable voting situations is the plurality rule when the size of the manipulating coalition is unrestricted and the Borda rule for small manipulating coalitions.

Keywords Manipulation · positional voting rules · Impartial Anonymous Culture

1 Introduction

Since the seminal contributions of Gibbard (1973) and Satterthwaite (1975), it is known that all reasonable voting rules are vulnerable to strategic voting. Pritchard and Wilson (2007) consider the problem of coalitional manipulation of positional voting rules (PVRs) in three-candidate elections with a moderate number of voters. For a broad analysis of a panel of strategic voting aspects, they present in terms of linear equations and inequalities a characterization of unstable voting situations (or anonymous profiles of individual preferences) at which a PVR is manipulable by a coalition - at those voting situations some nonempty subset of voters (a coalition) could misrepresent their actual preferences to favor the election of an outcome that they prefer to the outcome that would have resulted if they had voted sincerely. In this paper, a similar methodological approach is undertaken for large electorates and the results obtained generalize previous studies with the same settings.

The large electorate assumption is a framework that has been extensively used in the literature. See for example Lepelley and Valognes (2003) on the impact of social homogeneity on the manipulability of voting rules, Cervone et al. (2005) on the probability that a PVR selects the Condorcet winner or Diss and Gehrlein (2012) on Borda’s Paradox with
PVRs. Particularly, when algebraic computations with integer variables fail or become intractable, considering large electorates is unavoidable when one aims to study how frequent the occurrences of some electoral outcomes are. Moreover, almost all existing results on the manipulation of voting rules reflect the fact that results obtained for large electorates agree with exact results derived from complete enumerations or analytic representations as soon as the size of the electorate exceeds a few hundred voters; see, for example, Lepelley and Mbih (1987, 1994, 1997), Favardin et al. (2002) or more recently Pritchard and Wilson (2007).

Existing works on the manipulation of voting rules have raised a multitude of questions among which the following ones:

(Q1) Given a voting rule, what are the conditions under which the rule is vulnerable to strategic voting manipulation at a given profile (a configuration of individual preferences)? This is generally the preliminary but very important step for any further investigation.

(Q2) Given a voting rule, how frequent are the opportunities for strategic voting manipulation? As observed by Lepelley and Mbih (1994), this is a question of great importance for, if situations in which strategic voting manipulation can take place are rare, then the Gibbard-Satterthwaite result has only a limited interest, at least for that rule.

(Q3) How do some voting rules within a given class rank according to a given criterion of manipulability? For this purpose, several criteria or measures of manipulability have been proposed and include both qualitative and quantitative aspects; see Pritchard and Wilson (2007) for detailed references on this issue.

Peters et al. (2012) admirably addresses (Q1)-(Q3) within a class of voting rules that includes the approval rule together with some other related rules. In this paper, we are interested with the whole class of PVRs in three-candidate elections. A PVR with three candidates is supported by a scoring vector \((1, \lambda, 0)\) with \(0 \leq \lambda \leq 1\). A candidate receives a score of 1 point every time it is ranked as most preferred by a voter, \(\lambda\) points every time it is middle-ranked by a voter and no point otherwise. The winner is the candidate that receives the greatest total number of points from voters. As an answer to (Q1) in the specific case of PVRs, we provide a new characterization of unstable voting situations for any PVR. Our characterization only depends on the parameter \(\lambda\) of the rule and on the parameters that describe the total number of voters reporting the same type of strict rankings. We then use this advantage to extend many previous studies on manipulation from a limited set of selected rules to the entire class of PVRs.

A notable qualitative answer to (Q2) and (Q3) is suggested by Lepelley and Mbih (1994) and recently used by Pathak and Sönmez (2013). It relies on comparing the sets of profiles on which any two voting rules are manipulable. We refer to such comparisons as logical comparisons of manipulability of voting rules. More precisely, rule \(F\) is logically at least as manipulable as rule \(F'\) if all strategic voting opportunities under rule \(F'\) are also strategic voting opportunities under rule \(F\). Unfortunately, as already mentioned by Lepelley and Mbih (1994), many usual rules cannot be logically compared in the three-candidate elections. This is particularly the case of PVRs in consideration here. We show that there is no logical comparison of manipulability between any two PVRs. We then turn to a quantitative approach to compare the vulnerability of PVRs to strategic voting manipulation with the advantage that it always allows a full classification of PVRs according to their respective vulnerabilities to coalitional manipulation given a probability distribution over the set of possible profiles.

Several probability models support earlier results; see Gehrlein and Lepelley (2011) for a recent round up of most used probability distributions; or Smith (1999), Aleskerov et al.
(2011a, 2011b, 2012) and Gerhein et al. (2013) for several other settings on manipulability measures of voting rules. We focus our attention on the well-known Impartial Anonymous Culture (IAC) assumption introduced by Gehrlein and Fishburn (1976) and widely used by many other authors; see Lepelley and Mbih (1987, 1994), Huang and Chua (2000), Cer- vone et al. (2005) among others. It amounts to assuming that voters receive equal treatment and that all voting situations are equally likely to occur. From the characterization provided, we derive for three-candidate elections the probability that any PVR is manipulable by a coalition of voters as a function of the weight $\lambda$ that defines the scoring vector. As in Cervone et al. (2005) or in Diss and Gehrlein (2012), computations involve computing the 5-dimensional volume of some polytopes. Some known exact results on the manipulability of PVRs are due to Lepelley and Mbih (1987) for the plurality rule ($\lambda = 0$); Lepelley and Mbih (1994) for the antiplurality rule ($\lambda = 1$) or to Wilson and Pritchard (2007) for the Borda rule ($\lambda = 0.5$). In this paper, all these results on (simple) PVRs are recovered and generalized to the whole class of PVRs, that is for any weight $\lambda$ with $0 \leq \lambda \leq 1$.

Another aspect of manipulability raised by Pritchard and Wilson (2007) is the determination of the maximal minimum size of a manipulating coalition at a voting situation under a given PVR. Using complete computer enumerations, the authors observe for each example they consider that the minimum size of a manipulating coalition is always at most half the size of the electorate. We show in our settings that this conjecture is valid for three-candidate elections and for any PVR. Furthermore, we determine the maximal minimum size of a manipulating coalition as a function of the weight $\lambda$. This is also an important indicator as it informs about how hard it is to assemble and coordinate a manipulating coalition. Intuitively, given a voting rule, the greater the maximal minimum size of a manipulating coalition at a voting situation, the greater the effort to achieve a manipulation; for more details on this issue, see Reyhani et al. (2010). To put the influence of the size of the manipulating coalition into perspective, we compute the vulnerability of PVRs to strategic voting when the size of the manipulating coalition is less than or equal to a given value. The plurality rule (globally) minimizes the frequency of unstable voting situations when the size of the manipulating coalition is unrestricted while the Borda rule holds this trophy for small manipulating coalitions.

Before we continue, here is the overview of some key points of our contribution to the study of the manipulation of PVRs for three-candidate elections with large electorates:

1. Proofs are provided to validate the impossibility of logical comparisons between any pair of PVRs as earlier conjectured by Lepelley and Mbih (1994) with usual rules: that is when $\lambda \in \{0, \frac{1}{2}, 1\}$.
2. Given a PVR, a characterization of voting situations at which a manipulation may occur is obtained and an exact formula to derive the probability of manipulation for all PVRs is determined under IAC. These results are valid for all PVRs and contrast with previous results based on selected usual PVRs. Moreover, new results for several PVRs including the three usual PVRs are reported when only manipulation by coalitions with restricted sizes is considered$^1$.
3. The maximal minimum size of a manipulating coalition is obtained. This can be viewed as a response for large electorate to a conjecture by Wilson and Pritchard (2007) stating that the maximal minimum size of a manipulating coalition is at most half of the electorate.

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$^1$ Wilson and Pritchard (2007) present similar results for a list of six PVRs with less than 13 voters and also for 150 voters. These results are completed here with their corresponding values for large electorates.
Then to derive the outcome, it is sufficient to provide the 6-tuple within the same profile does not affect the outcome of a PVR: such a PVR is anonymous. Corresponding rule denoted by rule: that is the way ties are broken. For illustration, ties are alphabetically broken and the score. Each PVR is completely determined by its scoring vector and by its tie-breaking for that states the total number of points that each PVR in Section 3.4; and (ii) the probability that some PVRs are manipulable by a coalition of size less than or equal to a given fraction \( p \) of the electorate is evaluated to study the vulnerability of PVRs to manipulation by small coalitions.

2 Notation and definitions

Consider any process where \( n \geq 2 \) individuals or voters have to select a unique alternative out of a set \( A = \{a, b, c\} \) of three candidates. Assume that individual preferences over alternatives are linear orders (complete, antisymmetric and transitive binary relations) on \( A \). Then the six possible rankings are the followings:

\[
R_1 : abc \quad R_2 : acb \quad R_3 : bac \quad R_4 : bca \quad R_5 : cab \quad R_6 : cba. 
\]

A preference profile on \( A \) (a profile for short) is an \( n \)-tuple \( R^N = (R^1, R^2, \ldots, R^n) \) of \( n \) individual rankings where the \( i^{th} \) component \( R^i \) specifies voter \( i \)'s ordering on \( A \). Denote by \( L^N \), the set of all profiles. Then a social choice function or a voting rule is any mapping that associates any profile with a single alternative. In this paper, we are interested in PVRs described above also known as scoring voting rules. Given a profile \( R^N \) and a scoring vector \( w_\lambda = (1, \lambda, 0) \) with \( 0 \leq \lambda \leq 1 \), let \( S(u, \lambda, R^N) \) be the score of \( u \in A \) at \( R^N \in L^N \); note that \( S(u, \lambda, R^N) \) is the total number of points that \( u \) receives from voters. Then a voting rule \( F \) is a PVR supported by the scoring vector \( w_\lambda \) if for any profile \( R^N \), \( F(R^N) = u \) only if \( S(u, \lambda, R^N) \geq S(v, \lambda, R^N) \), for all \( v \in A \). In other words, the winner by a PVR is selected among the alternatives with the greatest score. Well-known representatives of PVRs are the plurality rule for \( w_0 = (1, 0, 0) \), the Borda rule for \( w_1 = (1, \frac{1}{2}, 0) \) and the antiplurality rule for \( w_1 = (1, 1, 0) \).

Ties may occur at a profile: two or more alternatives come first by recording the greatest score. Each PVR is completely determined by its scoring vector and by its tie-breaking rule: that is the way ties are broken. For illustration, ties are alphabetically broken and the corresponding rule denoted by \( F_9 \). By so doing, any permutation of individual preferences within the same profile does not affect the outcome of a PVR: such a PVR is anonymous. Then to derive the outcome, it is sufficient to provide the 6-tuple \( s = (n_1, n_2, n_3, n_4, n_5, n_6) \) that states the total number \( n_j \) of individuals reporting the \( j^{th} \) ranking \( R_j \) for each of the six rankings at (1). Any 6-tuple \( s = (n_1, n_2, n_3, n_4, n_5, n_6) \) of nonnegative integers such that \( \sum_{j=1}^{6} n_j = n \) will be called a voting situation.

From now on, we only consider the case of large electorates, that is when the size \( n \) of the electorate is sufficiently large (or tends to infinity), and we pose \( x_j = \frac{n_j}{n} \), \( j = 1, 2, \ldots, 6 \). Any voting situation \((n_1, n_2, n_3, n_4, n_5, n_6)\) will now be rewritten as \((x_1, x_2, x_3, x_4, x_5, x_6)\) such that

\[
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = \frac{n}{n} = 1
\]
where \( x_j \geq 0 \) is the proportion if voters reporting the \( j \)th ranking for \( j = 1, 2, \ldots, 6 \). The set \( S \) of all voting situations is then 5-dimensional.

Given a profile of individual preferences, some voters may be interested to know whether it is the best action to sincerely report their true preferences assuming that other voters are sincere. A voting rule is subject to strategic voting manipulation whenever it is possible that profiles might exist for which some nonempty subset of voters (a coalition) could misrepresent their actual preferences in an election to thereby obtain an outcome that they prefer to the outcome that would have resulted if they had voted sincerely. Such profiles are called unstable profiles. In other words, a profile is (coalitionally) unstable if it is not a strong Nash equilibrium.

More formally, given a profile \( R^N \) and a coalition \( S \), we write \( R^N = (R^S, R^-S) \) where \( R^S \) and \( R^-S \) stand respectively for the preference profile of individuals in \( S \) and the preference profile of individuals out of \( S \). Moreover \((Q^S, R^-S)\) is the profile obtained from \( R^N \) by replacing \( Q^i \) by \( R^i \) for each member \( i \) of \( S \). A voting rule \( F \) is manipulable at a given profile \( R^N \) if there exists a coalition \( S \) and an \( S \)-profile \( Q^S \) such that each member of \( S \) strictly prefers \( F(Q^S, R^-S) \) to \( F(R^N) \). In such a case, we say that \( F \) is manipulable from \( R^N \) to \( (Q^S, R^-S) \) by \( S \). \( F(R^N) \) is the sincere winner and \( F(Q^S, R^-S) \) is a strategic winner. Given two alternatives \( u \) and \( v \), we denote by: (i) \( M_u(F) \) the set of all voting situations at which \( F \) is manipulable and \( u \) is the strict winner (\( u \) wins without any tie); and (ii) \( M_{u,v}(F) \) the set of all voting situations at which \( u \) is the strict winner and \( v \) is a strategic winner (a strategic voting manipulation may occur in favor of \( v \)). Clearly

\[
M(F) = M_u(F) \cup M_{u,v}(F) \cup M_v(F)
\]

is the set of all voting situations at which there is a strict winner and \( F \) is manipulable by a coalition.

3 Manipulation of PVRs

In this section, our aim is to provide some answers to (Q1)-(Q3) within the class of PVRs.

3.1 Characterization of unstable voting situations

Consider a weight \( \lambda \in [0, 1] \) and a voting situation \( x = (x_1, x_2, x_3, x_4, x_5, x_6) \). We are interested in whether \( F_\lambda \) is manipulable by some coalition at \( x \), and if so, what are constraints that describe such voting situations. Without loss of generality, suppose that \( a \) is the strict winner under \( F_\lambda \). This is equivalent to \( S(b, \lambda, x) - S(a, \lambda, x) < 0 \):

\[
(\lambda - 1)x_1 - x_2 + (1 - \lambda)x_3 + x_4 - \lambda x_5 + \lambda x_6 < 0
\]

and \( S(c, \lambda, x) - S(a, \lambda, x) < 0 \):

\[
-x_1 + (\lambda - 1)x_2 - \lambda x_3 + \lambda x_4 + (1 - \lambda)x_5 + x_6 < 0
\]

Assume that \( F_\lambda \) is manipulable at \( x \) and that \( b \) is a strategic strict winner (\( b \) is the strict winner after a manipulation occurs). Only voters of type \( cba \) or \( bac \) are involved, voters of \( bca \) type benefit from this action but are not active since they are actually contributing the
maximum for \( b \) and nothing for \( a \). For such a manipulation to take place, one only needs a proportion \( y_1 \) of \( bac \) voters to report strategically \( bca \) while an appropriate proportion \( y_2 \) of \( cba \) voters report \( bca \) in such a way that \( b \) is the (strict) winner at the new profile now represented by the manipulated voting situation

\[
x' = (x_1, x_2, x_3 - y_1, x_4 + y_1, x_5, x_6 - y_2).
\]

This amounts to \( S(a, \lambda, x') - S(b, \lambda, x') < 0 \):

\[
(1 - \lambda)x_1 + x_2 - (1 - \lambda)x_3 - x_4 + \lambda x_5 - \lambda x_6 - \lambda y_1 - (1 - \lambda)y_2 < 0
\]

and \( S(c, \lambda, x') - S(b, \lambda, x') < 0 \):

\[
-\lambda x_1 + \lambda x_2 - x_3 - (1 - \lambda)x_4 + x_5 + (1 - \lambda)x_6 + \lambda y_1 - 2(1 - \lambda)y_2 < 0
\]

where

\[
0 \leq y_1 \leq x_3
\]

and

\[
0 \leq y_2 \leq x_6.
\]

Since \( 1 - \lambda \geq 0 \), (7) and (8) still hold when we set \( y_2 = x_6 \). Thus (7) and (8) become

\[
(1 - \lambda)x_1 + x_2 - (1 - \lambda)x_3 - x_4 + \lambda x_5 - \lambda x_6 - \lambda y_1 < 0
\]

and

\[
-\lambda x_1 + \lambda x_2 - x_3 - (1 - \lambda)x_4 + x_5 + (1 - \lambda)x_6 + \lambda y_1 < 0
\]

Note that for \( \lambda = 0 \), (11) and (12) simply become:

\[
x_1 + x_2 - x_3 - x_4 - x_6 < 0 \text{ and } -x_3 - x_4 + x_5 - x_6 < 0
\]

Now for \( \lambda > 0 \), rewriting (11) and (12) we obtain

\[
y_1 > \frac{(1 - \lambda)x_1 + x_2 + (\lambda - 1)x_3 - x_4 + \lambda x_5 - x_6}{\lambda}
\]

\[
y_1 < \frac{\lambda x_1 - \lambda x_2 + x_3 + (1 - \lambda)x_4 - x_5 + (1 - \lambda)x_6}{\lambda}
\]

Then for \( y_1 \) to exist, it is, by comparing bounds of \( y_1 \) from (9), (14) and (15), necessary and sufficient that

\[
-\lambda x_1 + \lambda x_2 - x_3 - (1 - \lambda)x_4 + x_5 + (1 - \lambda)x_6 < 0
\]

\[
(1 - \lambda)x_1 + x_2 - x_3 - x_4 + \lambda x_5 - x_6 < 0
\]

\[
(1 - 2\lambda)x_1 + (1 + \lambda)(x_2 + x_5) + (\lambda - 2)(x_3 + x_4 + x_6) < 0
\]

Note that (16) follows from the first inequality at (9) and (15), (17) comes from the second inequality at (9) and (14) while (18) is deduced from (14) and (15). Now by taking the linear combination\(^2\) \( \frac{1}{\lambda}(5) + \frac{1}{\lambda^2}(18) + \frac{1}{\lambda(1 - \lambda)}(-x_6 \leq 0) \), one obtains (16). Hence (16) is redundant. By canceling this redundant constraint, we conclude that

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\(^2\) Multiply the right hand side and the left hand side of each inequality by the corresponding coefficient.
Proposition 1 Let $\lambda \in [0, 1]$ and $x \in \mathcal{X}$. Assume that $a$ is the strict winner under $F_\lambda$. Then $F_\lambda$ is manipulable at $x$ and $b$ is a strict strategic winner (or equivalently $x \in M_{ab}(F_\lambda)$) if and only if $x$ satisfies (4), (5), (17) and (18).

Note that taking into account ties and a tie-breaking rule will only affect the set of constraints that characterize $M_{ab}(F_\lambda)$ by changing some strict inequalities ($< 0$) to their corresponding larger forms ($\leq 0$). From $M_{ab}(F_\lambda)$ to $M_{ac}(F_\lambda)$, one only needs to apply the transposition $b \leftrightarrow c$ (interchanging $b$ and $c$). This amounts to applying $x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_5$ and $x_4 \leftrightarrow x_6$ on constraints that support $M_{ab}(F_\lambda)$. The set $M_{ab}(F_\lambda) \cap M_{ac}(F_\lambda)$ is characterized by the collection of all constraints from $M_{ab}(F_\lambda)$ and $M_{ac}(F_\lambda)$.

3.2 Logical comparison of the manipulability of voting rules

Let $\lambda, \gamma \in [0, 1]$ with $\lambda \neq \gamma$. In this section, we aim to prove that there exists no logical comparison of manipulability between $F_\lambda$ and $F_\gamma$. We simply show that for each possible configuration, $M(F_\lambda)$ is not a subset of $M(F_\gamma)$. Note that $M_\lambda(F_\lambda) = \lambda \cup M_{ac}(F_\lambda)$.

Then, to prove that $M(F_\lambda) \not\subset M(F_\gamma)$, it is sufficient to construct some voting situations $s$ such that $s \in M_{ab}(F_\lambda) \cap M_{ac}(F_\lambda)$. To do this, we use the characterization of unstable voting situations under $F_\lambda$. Concretely both $M_{ab}(F_\lambda)$ and $M_{ac}(F_\lambda)$ are polytopes as shown in Proposition 1. We then explore the set of vertices that support $M_{ab}(F_\lambda)$ to find an appropriate combination of some selected vertices.

To facilitate the presentation, we now rewrite constraints in matrix form as $L_1, L_\lambda x < 0, \forall j \in \{1, 2, 3, 4\}$ where $L_1, L_\lambda = (-1, -1, -1, -1, -1, -1, -1, -1, -1)$, $L_2, L_\lambda = (1, -1, -1, -1, -1, -1, -1, -1, -1, -1)$, $L_3, L_\lambda = (1 - 2\lambda, 1 + \lambda, -2, -2 + \lambda, 1 + \lambda, -2, -2 + \lambda)$ and $L_4, L_\lambda = (1 - \lambda, 1 - \lambda, 1 - \lambda, 1 - \lambda)$ while $x$ is rewritten as a column vector. With these new settings, $x \in M_{ab}(\lambda)$ if and only if $L_1, L_\lambda x < 0, \forall j \in \{1, 2, 3, 4\}$ and $x \in M_{ac}(\lambda)$ if and only if $L_1, L_\lambda x < 0, \forall j \in \{1, 2, 3, 4\}$ where $\lambda = (x_1, x_2, x_3, x_4)$ corresponds to the voting situation obtained by permuting $b$ and $c$ in individual preferences. Before we continue, note that $L_1, L_\lambda x = L_2, L_\lambda x$ and $L_2, L_\lambda x = L_1, L_\lambda x$.

Proposition 2 Given a weight $\lambda$ such that $0 \leq \lambda \leq \frac{1}{2}$, there is no other weight $\gamma \in [0, 1]$ such that $F_\lambda$ is logically less manipulable than $F_\gamma$.

Proof Suppose that $\lambda < \gamma$. Then let $x = \left(\frac{1}{27}, \frac{5\lambda - 19}{27}, \frac{2}{27}, 0, \frac{2\lambda}{27}, -\frac{11\lambda - 13}{27}\right)$ and $x_{\lambda} = \alpha x + (1 - \alpha) y$ for any $\alpha \in [0, 1]$. These voting situations satisfy

$$
L_1, L_\lambda x = \frac{1}{27} \left(\frac{6x + 20\lambda - 7\lambda^2}{2 - \lambda}\right) < 0 \\
L_2, L_\lambda x = \frac{1}{27} \left(\frac{9 - 14\lambda + 7\lambda^2}{2 - \lambda}\right) < 0 \\
L_3, L_\lambda x = \frac{1}{27} \left(\frac{9 - 8\lambda + 3\lambda^2}{2 - \lambda}\right) < 0 \\
L_4, L_\lambda x = \frac{1}{27} \left(\frac{3\lambda^2 + 2}{2 - \lambda}\right) < 0
$$

Therefore $L_1, L_\lambda x < 0, \forall j \in \{1, 2, 3, 4\}$. Hence $x_{\lambda} \in M_{ab}(\lambda)$. We now show that for some small values of $\alpha, x_{\lambda} \notin M(F_\gamma)$. To see this, observe that

$$
L_1, y = \frac{(\gamma - \lambda)(\gamma - \lambda)}{3(\lambda - 1)} < 0 \\
L_2, y = \frac{(2 - \lambda)(\gamma - \lambda)}{3(\lambda - 1)} < 0 \\
L_3, y = \frac{(\lambda - \gamma)}{3(\lambda - 1)} < 0 \\
L_4, y = \frac{(\gamma - \lambda)}{3(\lambda - 1)} < 0
$$

Therefore $L_1, y < 0$ and $L_2, y < 0$. Hence $x_{\lambda} \notin M(F_\gamma)$. To see this, observe that

$$
L_1, x = \frac{(\gamma - \lambda)(\gamma - \lambda)}{2 - \lambda} < 0 \\
L_2, x = \frac{(\gamma - \lambda)(\gamma - \lambda)}{2 - \lambda} < 0
$$

Therefore $L_1, x < 0$ and $L_2, x < 0$. Hence $x_{\lambda} \notin M(F_\gamma)$.
Therefore, for \( \alpha \) sufficiently small, \( L_{1,\gamma} x_\alpha < 0 \), \( L_{2,\gamma} x_\alpha < 0 \), \( L_{3,\gamma} x_\alpha > 0 \) and \( L_{3,\gamma} \tilde{x}_\alpha > 0 \). For such values of \( \alpha \), \( F_\gamma (x_\alpha) = F_\gamma (\tilde{x}_\alpha) = a \) and no manipulation can occur either in favor of \( b \), or in favor of \( c \). That is \( x_\alpha \in M_{a,b} (\lambda) \subseteq M (F_\gamma) \) and \( x_\alpha \notin M (F_\gamma) \).

Now suppose that \( \lambda > \gamma \). We hold \( x = \left( \frac{1}{27}, \frac{2}{27}, \frac{7}{27}, 0, \frac{1}{27} \right) \) and renew \( y = \left( \frac{2}{27}, 0, 0, 0, \frac{2(1-\lambda)}{2-\lambda} \right) \). Then

\[
L_{1,\lambda} y = \frac{2\lambda}{2-\lambda} < 0 \quad L_{2,\lambda} y = \frac{2(\lambda-1)^2}{2-\lambda} < 0 \quad L_{3,\lambda} y = \frac{2(1-\lambda^2)}{2-\lambda} < 0 \quad L_{4,\lambda} y = 0 \quad (22)
\]

Therefore \( L_{j,\lambda} x_\alpha < 0 \), \( \forall j \in \{1, 2, 3, 4\} \). Hence \( x_\alpha \in M_{a,b} (\lambda) \). Moreover,

\[
L_{1,\gamma} y = \frac{2\lambda}{2-\lambda} < 0 \quad L_{2,\gamma} y = \frac{2(1-2\gamma+\lambda\gamma)}{2-\lambda} < 0 \quad L_{4,\gamma} y = \frac{2(\lambda-\gamma)}{2-\lambda} > 0 \quad L_{4,\gamma} \gamma = -L_{1,\gamma} y \quad (23)
\]

Thus, for \( \alpha \) sufficiently small, \( L_{1,\gamma} x_\alpha < 0 \), \( L_{2,\gamma} x_\alpha < 0 \), \( L_{3,\gamma} x_\alpha > 0 \) and \( L_{3,\gamma} \tilde{x}_\alpha > 0 \). For such values of \( \alpha \), \( F_\gamma (x_\alpha) = F_\gamma (\tilde{x}_\alpha) = a \) and no manipulation can occur either in favor of \( b \), or in favor of \( c \). That is \( x_\alpha \in M_{a,b} (\lambda) \) and \( x_\alpha \notin M (F_\gamma) \).

In both cases, \( \gamma > \lambda \) or \( \gamma < \lambda \), it comes out that \( M (F_\gamma) \nsubseteq M (F_\gamma) \).

Using very similar arguments for the case \( \frac{1}{2} < \lambda \leq 1 \), we also prove the following:

**Proposition 3** Given a weight \( \lambda \) such that \( \frac{1}{2} < \lambda \leq 1 \), there is no other weight \( \gamma \in [0, 1] \) such that \( F_\lambda \) is logically less manipulable than \( F_\gamma \).

**Proof** See Appendix A.

For each possible configuration of \( \lambda \) and \( \gamma \), it is shown that there exist some voting situations of the form \( x_\alpha \) such that \( x_\alpha \in M (F_\lambda) \) and \( x_\alpha \notin M (F_\gamma) \). In conclusion, merging Proposition 2 and Proposition 3, one derives the following:

**Proposition 4** There is no logical comparison of manipulability between any two PVRs for three-candidate elections.

### 3.3 The likelihood of unstable voting situations

To evaluate how often a voting phenomenon occurs, an approach that has been intensively used in the literature consists in assuming that all voting situations are equally likely to be observed. This probability distribution over the set of all voting situations is known as the Impartial Anonymous Culture (IAC) assumption. That is for example the case with Gehrlein and Fishburn (1976) on the transitivity of the majority rule; Gehrlein (1982) on the Condorcet efficiency of constant scoring rules; Lepelley and Mbih (1987, 1994, 1996) or Wilson and Pritchard (2007) on the manipulability of usual positional rules; and Lepelley et al. (1996) on the likelihood of monotonicity paradoxes. More references are provided in the book by Gehrlein and Lepelley (2011). For a discussion of these hypotheses, see Regenwetter, Grofman, Marley and Tsetlin (2006).

Concretely, as \( n \) tends to infinity the limit probability that a PVR \( F \) is coalitionally manipulable is computed as

\[
\text{vol} (F, \text{IAC}) = \frac{\text{vol} (M (F))}{\text{vol} (\mathcal{F})} \quad (24)
\]
where \( M(F) \) is the set of all voting situations at which \( F \) is manipulable and \( \text{vol}(D) \) stands for the 5-dimensional volume of a given subset \( D \) of \( \mathcal{S} \). Note that the set of all voting situations at which a tie occurs is a union of some \( d \)-dimensional subsets of \( \mathcal{S} \) with \( d < 5 \), thus the measure \( \text{vol}(F,\text{IAC}) \) does not depend on the chosen tie-breaking rule. Our concern in this section is the determination of \( \text{vol}(F,\lambda,\text{IAC}) \) as a function of the weight \( \lambda \), with \( 0 \leq \lambda \leq 1 \).

Note that

\[
M_a(\lambda) = M_{a,b}(\lambda) \cup M_{a,c}(\lambda).
\]

Then

\[
\text{vol}(M_a(\lambda)) = 2\text{vol}(M_{a,b}(\lambda)) - \text{vol}(M_{a,b}(\lambda) \cap M_{a,c}(\lambda))
\]

and since \( M_a(\lambda), M_b(\lambda) \) and \( M_c(\lambda) \) are symmetric (moving from one set to another, one simply permutes two alternatives),

\[
\text{vol}(M(F_\lambda)) = 3\text{vol}(M_a(\lambda)).
\]

Several methods or algorithms are available on volume computations for polytopes; see for example B"ueler et al. (2000) or Lawrence (1991); see also Lepelley et al. (2008) for recent developments, Cervone et al. (2005) or Diss and Gehrlein (2012) for practical illustrations with parameterized polytopes. In this paper we use, for all volume computations, a triangulation method derived from the well known Cohen and Hickey algorithm of triangulating a polytope (Cohen and Hickey, 1979). Let \( P \) be a given \( d \)-dimensional polytope described by some non redundant linear inequalities \( E_j : c_jy \leq b_j, j = 1, 2, \ldots, m \). Each facet \( F_j \) of \( P \) corresponds to at most one equation \( c_jy = b_j \) with \( j \in \{1, 2, \ldots, m\} \). Each vertex can then be attached to the subset of facets it belongs to. Choosing a vertex, said \( v_1 \), a dissection of \( P \) is obtained by considering all pyramids \( v_1F_j \) with apex \( v_1 \) and bases \( F_j \) such that \( v_1 \) is out of \( F_j \). This operation is then applied recursively to find a triangulation of \( P \) into simplices, each containing \( d + 1 \) points that are affine independent. Finally the volume of \( P \) is the sum of the volumes of simplices obtained in its triangulation using the following formula of the \( d \)-dimensional volume of a simplex \( \Delta(a_0, a_1, \ldots, a_d) \):

\[
\text{vol}(\Delta(a_0, a_1, \ldots, a_d)) = \frac{|\text{det}(a_1 - a_0, a_2 - a_0, \ldots, a_d - a_0)|}{d!} \text{vol}_0
\]

where each \( a_j \) is a vertex of \( \Delta(a_0, a_1, \ldots, a_d) \), the operator \( \text{det} \) stands for the determinant and \( \text{vol}_0 \) is a constant that depends on the cartesian coordinate system used for vertices. Since the measure at (24) is a ratio, one can simply set \( \text{vol}_0 = 1 \).

Using this method with more details and illustrations relegated to Appendix B, we compute the volumes of \( M_{a,b}(\lambda) \) and \( M_{a,c}(\lambda) \) to derive from (24), (26) and (27) the following result:

**Proposition 5** Consider \( \lambda \in [0, 1] \). Then as \( n \) tends to infinity, the probability that any resolute version of the scoring rule associated with \( (1, \lambda, 0) \) is coalitionally unstable is as follows:

\[
\text{val}(F_\lambda,\text{IAC}) = \frac{56 \lambda^{18} - 1080 \lambda^{17} + 9136 \lambda^{16} - 45040 \lambda^{15} + 144536 \lambda^{14} - 316048 \lambda^{13} - 470044 \lambda^{12}}{324 (\lambda^2 - 2 \lambda + 2)^2 (1 + \lambda)^2 (4 - \lambda)^2 (1 - \lambda)^2 (2 - \lambda)^3}
\]

Using this method with more details and illustrations relegated to Appendix B, we compute the volumes of \( M_{a,b}(\lambda) \) and \( M_{a,c}(\lambda) \) to derive from (24), (26) and (27) the following result:
captures the propensity of each PVR to allow coalitional strategic voting; thus one may

\[
\frac{-507624\lambda^5 - 6692\lambda^4 - 23272\lambda^3 + 220752\lambda^2 - 184320\lambda + 48384}{324(\lambda^2 - 2\lambda + 2)^2(1 + \lambda)^2(4 - \lambda)^2(1 - \lambda)^2(2 - \lambda)^3}
\]

For \(\frac{1}{2} \leq \lambda \leq 1\), \(vul(F_\lambda, IAC) = \)

\[
2(7\lambda^{16} - 162\lambda^{15} + 1667\lambda^{14} - 10204\lambda^{13} + 42082\lambda^{12} - 125884\lambda^{11} + 287678\lambda^{10})
\]

\[
\frac{81\lambda(\lambda^2 - 2\lambda + 2)^2(1 + \lambda)^2(4 - \lambda)^2(\lambda - 2)^3}{81\lambda(\lambda^2 - 2\lambda + 2)^2(1 + \lambda)^2(4 - \lambda)^2(\lambda - 2)^3}
\]

\[
+ 2(-521859\lambda^9 + 768955\lambda^8 - 924081\lambda^7 + 892792\lambda^6 - 677528\lambda^5 + 400860\lambda^4)
\]

\[
\frac{81\lambda(\lambda^2 - 2\lambda + 2)^2(1 + \lambda)^2(4 - \lambda)^2(\lambda - 2)^3}{81\lambda(\lambda^2 - 2\lambda + 2)^2(1 + \lambda)^2(4 - \lambda)^2(\lambda - 2)^3}
\]

\[
+ 2(-193308\lambda^3 + 81876\lambda^2 - 25952\lambda + 2368)
\]

\[
\frac{81\lambda(\lambda^2 - 2\lambda + 2)^2(1 + \lambda)^2(4 - \lambda)^2(\lambda - 2)^3}{81\lambda(\lambda^2 - 2\lambda + 2)^2(1 + \lambda)^2(4 - \lambda)^2(\lambda - 2)^3}
\]

Table 1 presents some computed values of \(vul(F_\lambda, IAC)\). We recover earlier results for:

(i) the plurality rule by Lepelley and Mbih (1987) (29.12% of unstable voting situations); (ii)

the antiplurality rule by Lepelley and Mbih (1994) (51.85% of unstable voting situations); and

(iii) the Borda rule by Wilson and Pritchard (2007) (50.25% of unstable voting situations).

Moreover, the general formula presented in Proposition 5 provides a clear answer to the

following question: what is the least (or the most) vulnerable PVR to strategic voting

manipulation with \(\lambda\) corresponding to \(\lambda = 0.5\) is known to be particularly vulnerable to strategic vot-

ing manipulation with \(\lambda = 0.5\) is known to be particularly vulnerable to strategic vot-

ing; (ii) \(vul(F_\lambda, IAC)\) increases from 0.2912 to 0.5556 as \(\lambda\) grows from 0 to \(\hat{\lambda}\); but decreases from 0.5556 to 0.5186 as \(\lambda\) grows from \(\hat{\lambda}\) to 1. Although the Borda rule (\(\lambda = 0.5\)) is known to be particularly vulnerable to strategic voting

manipulation with \(50.25\%\) of unstable voting situations, it is not the most manipulable

PVR. With respect to the global frequency of coalitional unstable voting situations for three-

candidate elections with large electorates, it seems that the plurality rule (\(\lambda = 0\)) is the least

vulnerable PVR with 29.12% of unstable voting situations while the most vulnerable PVR

corresponds to \(\lambda = \hat{\lambda} = 0.78739\) with 55.56% of unstable voting situations under the IAC

assumption.

Table 1 Vulnerability of positional voting rules to strategic voting under IAC

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(vul(F_\lambda, IAC))</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0.29167</td>
</tr>
<tr>
<td>0.1</td>
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</tr>
<tr>
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<tr>
<td>0.3</td>
<td>0.41319</td>
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<tr>
<td>0.4</td>
<td>0.46075</td>
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<tr>
<td>0.5</td>
<td>0.50247</td>
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<tr>
<td>0.6</td>
<td>0.53255</td>
</tr>
<tr>
<td>0.7</td>
<td>0.55036</td>
</tr>
<tr>
<td>0.8</td>
<td>0.55548</td>
</tr>
<tr>
<td>0.9</td>
<td>0.54581</td>
</tr>
<tr>
<td>1</td>
<td>0.51832</td>
</tr>
</tbody>
</table>

Although a full classification of PVRs is obtained here, it is worth noticing that it only
captures the propensity of each PVR to allow coalitional strategic voting; thus one may
still need to combine our conclusion with other criteria for more complete judgements. For example, we have seen in section 3.1 that members of a manipulating coalition have to vote in a specific way. Thus, for a successful manipulation, there is an effort of coordination that \( \text{vul} (F_\lambda, IAC) \) does not handle (even when the complete knowledge assumption is implicitly admitted). The next section lays emphasis on this aspect.

### 3.4 Maximal minimum size of manipulating coalitions

Given a PVR \( F_\lambda \), consider a voting situation \( x \) at which there is some coalition that can manipulate; in other words, \( x \in M(F_\lambda) \). Let us denote by \( P(\lambda, x) \) the minimal size of a manipulating coalition under \( F_\lambda \) at \( x \) where the size of a coalition is the proportion of the electorate it represents. In terms of effort for recruitment and coordination, manipulations by a minimum number of individuals are the cheapest. We are then interested in determining \( P^\ast (\lambda) \), the maximum minimum size of a manipulating coalition under \( F_\lambda \). That is

\[
P^\ast (\lambda) = \max_{x \in M(F_\lambda)} P(\lambda, x).
\]

If for example \( P^\ast (\lambda) = 0.05 \), then the conclusion is that we need at most to coordinate 5\% of the voters to be able to manipulate the rule \( F_\lambda \) whenever it is manipulable.

To evaluate \( P^\ast (\lambda) \), we assume without loss of generality that \( x \in M_{a,b}(F_\lambda) \) and that there exists at \( x \) a manipulating coalition the size of which is \( p \). To handle the influence this new parameter \( p \) on the set of constraints that define \( M_{a,b}(F_\lambda) \) and to capture its range, we reconsider the general analysis of manipulation circumstances as in Section 3.1. Since \( x \in M_{a,b}(F_\lambda) \), manipulation in favor of \( b \) by a coalition of size \( p \) occurs when an appropriate
proportion $y_1$ of bac individuals report $bca$ while an appropriate proportion $y_2$ of cba individuals report $bca$ (with $p = y_1 + y_2$). In this case, $x$, $y_1$ and $y_2$ satisfy (4), (5), (7), (8), (9) and (10). Since $p = y_1 + y_2$, (7) and (8) become

$$(1 - \lambda) x_1 + x_2 - (1 - \lambda) x_3 - x_4 + \lambda x_5 - \lambda x_6 + (1 - 2\lambda) y_1 - (1 - \lambda) p < 0$$

and

$$-\lambda x_1 + \lambda x_2 - x_3 - (1 - \lambda) x_4 + x_5 + (1 - \lambda) x_6 + (2 - \lambda) y_1 - 2(1 - \lambda) p < 0.$$  

For each of the cases we distinguish below, we first find by elimination the range of $y_1$ and thereafter the range of $p$. For this purpose, remember that $0 \leq y_1 \leq x_3$, $y_1 \leq p$ and $y_2 = p - y_1 \leq x_6$. To control the sign of each coefficient (of $y_1$ or $p$) encountered in the elimination process, we have to undertake two distinct treatments firstly for $0 < \lambda < \frac{1}{2}$ and secondly for $\frac{1}{2} < \lambda < 1$. We have also checked that the conclusion obtained with $0 < \lambda < \frac{1}{2}$ still holds for both $\lambda = 0$ and $\lambda = \frac{1}{2}$ while the result obtained for $\frac{1}{2} < \lambda < 1$ remains valid for both $\lambda = 1$ and $\lambda = \frac{1}{2}$.

Assume that $0 < \lambda < \frac{1}{2}$. Collecting all constraints that depend on $y_1$ shows that the lower bound of $y_1$ is $y_1^* = \max (0, p - x_6)$ and its upper bound is $y_1^* = \max (p, x_3, r, t)$ with

$$r = \frac{1 - \lambda}{1 - 2\lambda} (x_3 + p - s_1) - x_2 + x_4 - \lambda (x_5 - x_6)$$

and $t = \frac{\lambda (x_1 - s_2) + x_1 - s_3 + (1 - \lambda) (x_4 + 2p - x_6)}{2 - \lambda}$.  

We deduce that $r > 0$, $t > 0$, $p - x_6 \leq x_3$, $p - x_6 < r$ and $p - x_6 < t$. Finally the lower bound of $p$ is $P(\lambda, x) = \min (u_1(x), u_2(x), u_3(x), 0)$ where

$$u_1(x) = \frac{(1 - \lambda) x_1 + x_2 - (1 - \lambda) x_3 - x_4 + x_5 - (1 - \lambda) x_6}{1 - \lambda},$$

$$u_2(x) = \frac{(1 - \lambda) x_1 + x_2 - (1 - \lambda) x_3 - x_4 + x_5 + \lambda x_6}{1 - \lambda},$$

$$u_3(x) = \frac{\lambda x_1 + x_2 + x_3 + (1 - \lambda) x_4 + x_5 - (1 - \lambda) x_6}{2(1 - \lambda)}.$$  

Since $P(\lambda, x)$ is picked between $u_1(x), u_2(x), u_3(x)$ and 0, we conclude that

$$P^*(\lambda) = \max (u_1^*(x), u_2^*(x), u_3^*(x), 0)$$

where $u_j^*(x)$ is the upper bound solution to the following linear program

$$\max_{s.t. x \in M_{a,b}} u_j(x)$$

Constraints that describe $M_{a,b}(F_2)$ can be rewritten with positive slack variables as:

$$
\begin{align*}
-x_1 + (\lambda - 1) x_2 - \lambda x_3 + \lambda x_4 + (1 - \lambda) x_5 + x_6 + s_1 &= 0 \\
(\lambda - 1) x_1 - x_2 + (1 - \lambda) x_3 + x_4 - \lambda x_5 + \lambda x_6 + s_2 &= 0 \\
(1 - 2\lambda) x_1 + (1 + \lambda) (x_2 + x_3) - (2 - \lambda) (x_3 + x_4 + x_6) + s_3 &= 0 \\
(1 - \lambda) x_1 + x_2 - x_3 - x_4 + \lambda x_5 - x_6 + s_4 &= 0 \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - 1 &= 0
\end{align*}
$$

We then solve these five constraints for $x_6$, $x_2$, $x_1$, $x_3$ and $x_5$ to obtain

$$u_1(x) = \frac{1}{2} x_1 - \frac{x_2}{2} - \frac{x_3}{2} - \frac{x_4}{2} - \frac{x_5}{2} + \frac{\lambda}{2},$$

$$u_2(x) = \frac{1}{2} x_1 - \frac{x_2}{2} - \frac{x_3}{2} - \frac{x_4}{2} + \frac{x_5}{2} + \frac{\lambda}{2}.$$
Similarly, solving constraints above for \( x_2, x_5, x_6, s_2 \) and \( s_4 \) yields
\[
u_3(x) = \frac{1 + \lambda}{2} - \frac{3}{4} x_1 - \frac{3}{4} x_3 - \frac{1}{4 (1 - \lambda)} s_1 - \frac{\lambda}{4 (1 - \lambda)} s_3.
\] (37)

Note that each variable in \( u_j(x) \) is non-negative and is assigned a non-positive coefficient. Moreover, positive slack variables are involved. We then deduce that for all \( x \in M_{a,b}(F_\lambda) \),
\[
u_1(x) < \frac{1}{2}, \quad u_2(x) < \frac{1}{2}, \quad \text{and} \quad u_3(x) < \frac{1 + \lambda}{4} \leq \frac{1}{2}
\] (38)
and that \( P^*(\lambda) = \frac{1}{2} \) only on the boundary of the domain defined by \( M_{a,b}(F_\lambda) \). In conclusion,

**Proposition 6** Given that \( 0 \leq \lambda \leq \frac{1}{2} \), the maximum minimum size \( P^*(\lambda) \) of a manipulating coalition at any unstable voting situation under \( F_\lambda \) is at most \( \frac{1}{2} \).

For \( \frac{1}{2} < \lambda < 1 \), we perform a very similar analysis as above. The result is as follows:

**Proposition 7** Given that \( \frac{1}{2} \leq \lambda \leq 1 \), the maximum minimum size \( P^*(\lambda) \) of a manipulating coalition at any unstable voting situation under \( F_\lambda \) is at most \( \frac{2 - \lambda}{3} \).

**Proof** See Appendix D.

As earlier announced \( P^*(\lambda) \) can be viewed as the proportion of voters one needs at most to coordinate to be able to manipulate \( F_\lambda \) whenever it is manipulable. Thus the message from Proposition 6 and Proposition 7 is that to be able to manipulate \( F_\lambda \), one needs at most to coordinate 50% of the voters for each rule weighted between the plurality rule and the Borda rule; and this percentage decreases as an affine function of the weight from 50% to 33.33% when \( \lambda \) increases from \( \frac{1}{2} \) (Borda rule) to 1 (antiplurality rule). Therefore when one considers the maximum minimum size of a manipulating coalition as an indicator of how difficult it is to implement a manipulation, it seems that: (i) PVRs weighted between 0 and \( \frac{1}{2} \) share the same performance; and (ii) each PVR weighted between 0 and \( \frac{1}{2} \) outperforms any other PVR weighted between \( \frac{1}{2} \) and 1. Note that the plurality rule and the Borda rule are among frontrunners while the antiplurality rule lags behind on this issue.

### 3.5 Manipulation by coalitions of size less or equal to a given proportion

To check how far the size of the manipulating coalition influences the vulnerability of a rule, we now evaluate the probability \( \text{val}(p, F_\lambda, \text{IAC}) \) that the PVR \( F_\lambda \) is manipulable by a coalition of size less than or equal to a given \( p \). That is
\[
\text{val}(p, F_\lambda, \text{IAC}) = \frac{\text{vol}(M^p(F_\lambda))}{\text{vol}(\mathcal{Z})}
\] (39)

where \( M^p(F_\lambda) \) is the set of all voting situations at which \( F_\lambda \) is manipulable by a coalition of size less than or equal to \( p \). This is done by replacing in the analysis made in Section 3.4 the constraint \( y_1 + y_2 = p \) with \( y_1 + y_2 \leq p \). More precisely, \( F_\lambda \) is manipulable by a coalition of size less than or equal to \( p \) at a voting situation \( x \in M_{a,b}(F_\lambda) \) if and only if \( x \) satisfies (i) (4) and (5); and (ii) there exists two integers \( y_1 \) and \( y_2 \) such that the voting situation defined at (6) satisfies (7), (8), (9), (10) and
\[
y_1 + y_2 \leq p.
\] (40)
By following the same elimination process as done in Section 3.1, the set of constraints obtained by elimination of $y_1$ and $y_2$ characterizes the set $M^p_{a,b} (F_\lambda) = M^p_{a,b} (F_\lambda) \cap M^p (F_\lambda)$ of all voting situations $x \in M_{a,b} (F_\lambda)$ at which $a$ is the strict winner and $F_\lambda$ is manipulable in favor of $b$ by some coalition of size less than or equal to $p$. In the same way, $M^p_{a,c} (F_\lambda) = M_{a,c} (F_\lambda) \cap M^p (F_\lambda)$ is the set of all voting situations $x \in M_{a,c} (F_\lambda)$ at which $a$ is the strict winner and $F_\lambda$ is manipulable in favor of $c$ by some coalition of size less than or equal to $p$.

As in (26) and (27),

$$vol (M^p (F_\lambda)) = 6vol \left( M^p_{a,b} (F_\lambda) \right) - 3vol \left( M^p_{a,b} (F_\lambda) \cap M^p_{a,c} (F_\lambda) \right).$$

(41)

For example, the reader can check that for the plurality rule $F_0$,

$$x \in M^p_{a,b} (F_0) \iff \begin{cases} -x_1 - x_2 + x_5 + x_6 < 0 \\ -x_1 - x_2 + x_3 + x_4 < 0 \\ x_1 + x_2 - x_3 - x_4 - x_6 < 0 \\ x_1 + x_2 - x_3 - x_4 - p < 0 \end{cases}$$

(42)

and

$$x \in M^p_{a,b} (F_0) \cap M^p_{a,c} (F_0) \iff \begin{cases} -x_1 - x_2 + x_5 + x_6 < 0 \\ -x_1 - x_2 + x_3 + x_4 < 0 \\ x_1 + x_2 - x_3 - x_4 - x_6 < 0 \\ x_1 + x_2 - x_3 - x_4 - p < 0 \\ x_1 + x_2 - x_5 - x_6 - p < 0 \\ x_1 + x_2 - x_4 - x_5 - x_6 < 0 \end{cases}$$

(43)

Recalling that each voting situation satisfies (2) and has non negative components, we compute $vol \left( M^p_{a,b} (F_\lambda) \right)$ and $vol \left( M^p_{a,b} (F_\lambda) \cap M^p_{a,c} (F_\lambda) \right)$ using the same triangulation method described in both Section 3.3 and Appendix B. For the plurality rule $F_0$, the Borda rule $F_{0,5}$ and the antiplurality rule $F_1$, manipulability results as functions of $p$ are reported in Appendix C and sketched in Figure 2. It appears that:

(i) The manipulability of each of the three usual rules increases as $p$ increases from 0 to a threshold $p^*$ (with $p^* = \frac{1}{2}$ for the antiplurality rule and $p^* = \frac{1}{5}$ for both the plurality rule and the Borda rule); and then becomes constant as $p$ increases from $p^*$ to 1. Note that the behavior of each rule for $p \in [p^*, 1]$ is in fact consistent with Proposition 6 and Proposition 7;

(ii) With respect to the manipulation by coalitions of size less than or equal to a given fraction of the voters, the antiplurality poorly performs as compared to the plurality rule and the Borda rule;

(iii) Although the plurality rule appears in Section 3.3 to be less manipulable than the Borda rule when we consider the possibility of manipulation by a coalition of any size, the Borda rule is now less manipulable than the plurality rule according to manipulation with small coalitions. More precisely, the Borda rule is less manipulable than the plurality rule as the size $p$ of the manipulating coalition increases from 0 to a threshold $\hat{p} \approx 0.1058$ while the plurality rule recovers its leading position as $p$ increases from $\hat{p}$ to 1.

More numerical results obtained are presented in Table 2 only for $p \in [0, \frac{1}{2}]$ since by Proposition 6 and Proposition 7, $vul (p, F_\lambda, IAC)$ does not depend on $p$ for all $p \in \left[ 0, \frac{1}{2} \right]$ and for all $\lambda \in [0,1]$; more precisely $vul (p, F_\lambda, IAC) = vul (F_\lambda, IAC)$ for $p \geq \frac{1}{2}$. As mentioned
above, we have determined $vul(p, F_λ, IAC)$ for $λ = \frac{k}{10}$ with $k \in \{0, 1, 2, \ldots, 10\}$. It appears that:

(a) As function of $p$, the curve of $vul(p, F_λ, IAC)$ given each value of $λ$ has the same shape as those sketched in Figure 2. More precisely, for each of the eleven values of $λ$ we consider, $vul(p, F_λ, IAC)$ increases as $p$ increases from 0 to a threshold $p^*$ and then becomes constant as $p$ increases from $p^*$ to 1 where $p^*$ is given by Proposition 6 and Proposition 7.

(b) In the upper part of Table 2, we first compute $vul(p, F_λ, IAC)$ for $p = \frac{k}{10}$ with $k \in \{1, 2, 3, 4, 5\}$. The probability $vul(p, F_λ, IAC)$ that $F_λ$ is manipulable by a coalition of size less than or equal to $p$ increases as $λ$ increases: (a1) from 0 to 0.8 for $p = 0.3, 0.4, 0.5$; (a2) from 0 to 0.9 for $p = 0.2$; and (a3) from 0 to 1 for $p = 0.1$. For these values, the rules that minimize $vul(p, F_λ, IAC)$ is the Borda rule for $p = 0.1$ and the plurality rule for $p = \frac{k}{10}$ with $k \in \{2, 3, 4, 5\}$.

(c) In the middle part of Table 2, we consider manipulation only by a small coalition of size less than or equal to $p = \frac{k}{10}$ with $k \in \{1, 2, \ldots, 9\}$. Results confirm the observation that appears for $p = 0.1$: when only manipulation by small coalitions is considered, the Borda rule minimizes $vul(p, F_λ, IAC)$ while the antiplurality maximizes it. As this conclusion is also confirmed in the lower part of Table 2 on the manipulation with very small coalitions, it constitutes our overall comment on the influence that the size of the manipulating coalition has on the vulnerability of a given PVR.

3 Exact representations for $vul(p, F_λ, IAC)$ with $λ = \frac{k}{10}$, $k = 1, 2, 3, 4, 6, 7, 8, 9$ are also piecewise defined functions in $p$ with numerous parts. All those results are available from the author upon simple request.
The IAC assumption. The plurality rule globally minimizes the vulnerability to coalitional manipulation under some manipulating coalition of size less than or equal to \( p \).

### Table 2 Probability that \( F_3 \) is manipulable by some manipulating coalition of size less than or equal to \( p \)

<table>
<thead>
<tr>
<th>Size ( p )</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
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<td>37.06</td>
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<table>
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<th>0.05</th>
<th>0.04</th>
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<td>17.31</td>
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<td>13.47</td>
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<td>12.67</td>
<td>12.11</td>
<td>15.00</td>
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</tr>
</tbody>
</table>

In an earlier result by Saari (1990), the Borda rule is also the most performing PVR with respect to the binary susceptibility measure, that is the proportion of voting situations at which a small coalition may reverse the relative ranking of two candidates. We consider voting rules (each outcome is a candidate) while Saari deals with voting procedures (each outcome is a possible ranking of candidates). Clearly, reversing the relative ranking of two candidates does not necessarily lead to a manipulation in our sense, unless the top ranked alternative in the initial profile is concerned. This may explain why the plurality rule outperforms the antiplurality rule here while a tie occurs between the two rules under the binary susceptibility measure.

### 4 Concluding remarks

This paper was concerned with the manipulability of scoring rules. Distinct aspects have been explored.

For three-candidate elections and for large electorates, the present paper provides the close form representation of the vulnerability of PVRs to coalitional manipulation under the IAC assumption. The plurality rule globally minimizes the vulnerability to coalitional
manipulation. This is no longer the case as soon as the size of the manipulating coalition is less than or equal to 10%. When only manipulation by small coalitions is considered, the occurrence of unstable voting situations is minimal under the Borda rule and maximal under the antiplurality rule.

To check the impact of the IAC assumption on our conclusions, the same analysis can be undertaken for any other reasonable probability distribution on profiles. For example, Pritchard and Wilson (2007) report two distinct orderings of voting rules alternatively under the IAC assumption and under the impartial culture (that is when all profiles are equally likely). For such initiatives, it will also be interesting to compare the performances of voting rules for small manipulating coalitions. Note that the maximum minimum size of the manipulating coalition does not depend on the probability distribution as long as all voting situations are observable, even with distinct probabilities.

The attempt to compare the manipulability of PVRs via set inclusion of their corresponding sets of unstable voting situations fails: there is no such comparison on any pair of PVRs for three-candidate elections. The question is whether this negative conclusion still holds for more than three candidates; this is still an open question. Lepelley (1995) shows that the hare rule (plurality with runoff) is logically least manipulable than the plurality rule. A further investigation can then be carried on to check whether this conclusion is still valid for any simple PVR and its iterative version.

Appendix

Appendix A: Proof of Proposition 3

Proof Suppose that \( \gamma < \lambda \). We use the same notation as in the proof of Proposition 2 to show that there is no other weight \( \gamma \in [0,1] \) such that \( F_{\lambda} \) is logically less manipulable than \( F_{\gamma} \).

Reset \( x = \left( \frac{5}{12}, 0, \frac{5}{12}, 0, \frac{1}{6}, 0 \right) \) and \( y = \left( \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0 \right) \). These voting situations satisfy

\[
\begin{align*}
L_{1,\lambda} x &= -\frac{2\lambda + 3}{12} < 0 & L_{1,\lambda} y &= -\frac{\lambda + 2 + 1}{\lambda - 2} \leq 0 \\
L_{2,\lambda} x &= -\frac{3}{8} < 0 & L_{2,\lambda} y &= 0 \\
L_{3,\lambda} x &= -\frac{1 + \lambda}{2} < 0 & L_{3,\lambda} y &= -\frac{1 + 2}{\lambda - 2} < 0 \\
L_{4,\lambda} x &= -\frac{1}{2} < 0 & L_{4,\lambda} y &= 0
\end{align*}
\]

Clearly, \( L_{j,\lambda} x_\alpha < 0 \), \( \forall j \in \{1, 2, 3, 4\} \) for all \( \alpha \in ]0, 1[ \). Thus \( x_\alpha \in M_{a,b}(\lambda) \). Furthermore,

\[
\begin{align*}
L_{1,\gamma} y &= \frac{1 + \gamma + \lambda}{\lambda - 2} < 0 & L_{2,\gamma} y &= \frac{\lambda - \gamma}{\lambda - 2} < 0 & L_{4,\gamma} y &= \frac{\lambda - \gamma}{\lambda - 2} > 0 & L_{4,\gamma} y &= \frac{\lambda}{\lambda - 2} > 0
\end{align*}
\]

Then for small values of \( \alpha \), \( L_{1,\gamma} x_\alpha < 0 \), \( L_{2,\gamma} x_\alpha < 0 \), \( L_{4,\gamma} x_\alpha > 0 \) and \( L_{4,\gamma} x_\alpha > 0 \). For such values of \( \alpha \), \( F_{\gamma}(x_\alpha) = F_{\gamma}(\hat{x}_\alpha) = a \) and no manipulation can occur nor in favor of \( b \); nor in favor of \( c \). Thus \( x_\alpha \in M(F_{\gamma}) \) and \( x_\alpha \notin M(F_{\lambda}) \).

Now suppose that \( \gamma > \lambda \). We hold \( x = \left( \frac{5}{12}, 0, \frac{5}{12}, 0, \frac{1}{6}, 0 \right) \) and \( y = \left( 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0 \right) \). Remember that \( L_{j,\lambda} x < 0 \), \( \forall j \in \{1, 2, 3, 4\} \) and note that

\[
\begin{align*}
L_{1,\lambda} y &= 2L_{2,\lambda} y = -\frac{2\lambda - 1}{3} < 0 & L_{3,\lambda} y &= 0 & L_{4,\lambda} y &= -\frac{1 - \lambda + 2}{3} < 0
\end{align*}
\]
Hence \( L_{\alpha, \lambda} x_{\alpha} < 0, \forall j \in \{1, 2, 3, 4\} \) for all \( \alpha \in \{0, 1\} \). Thus \( x_{\alpha} \in M_{a,b}(\lambda) \). To see that \( x_{\alpha} \notin M(F_j) \), note that for \( \gamma > \lambda > \frac{1}{2} \),

\[
\begin{align*}
L_{1, \gamma} y &= -\frac{\gamma + \gamma - 2}{3} < 0 \\
L_{2, \gamma} y &= \frac{\gamma - y + 1}{\gamma} < 0 \\
L_{3, \gamma} y &= \gamma - \lambda > 0 \\
L_{4, \gamma} y &= \lambda + \gamma - 1 > 0
\end{align*}
\]

Then for small values of \( \alpha, L_{1, \gamma} x_{\alpha} < 0, L_{2, \gamma} x_{\alpha} < 0, L_{3, \gamma} x_{\alpha} > 0 \) and \( L_{4, \gamma} x_{\alpha} > 0 \). For such values of \( \alpha, F_j(x_{\alpha}) = F_j(\tilde{x}_{\alpha}) = a \) and no manipulation can occur nor in favor of \( b \); nor in favor of \( c \). Thus there exists some values of \( \alpha \) such that \( x_{\alpha} \in M_{a,b}(\lambda) \) and \( x_{\alpha} \notin M(F_j) \).

Appendix B: More details on volume computations

Note that the set of constraints that completely describe \( M_{a,b}(\lambda) \) includes (4), (5), (17), (18) together with the six constraints \( x_j \geq 0, j = 1, 2, \ldots, 6 \) that guarantee the non-negativity of the six variables. Since each voting situation \( x \) satisfies (2), we simply identify \( x \) by the 5–tuple \( x \equiv (x_1, x_2, x_3, x_4, x_5) \) given that \( x_6 = 1 - (x_1 + x_2 + x_3 + x_4 + x_5) \) and \( x_1 + x_2 + x_3 + x_4 + x_5 \leq 1 \). Therefore by ruling out \( x_6 \), \( M_{a,b}(\lambda) \) is now determined by the following list of 10 linear inequalities without none being redundant:

\[
\begin{cases}
-2x_1 + (\lambda - 2)x_2 - (1 + \lambda)x_3 - (1 - \lambda)x_4 - \lambda x_5 \leq -1 & (I_1) \\
-x_1 - (1 + \lambda)x_2 + (1 - 2\lambda)x_3 + (1 - \lambda)x_4 - 2\lambda x_5 \leq -\lambda & (I_2) \\
(2 - \lambda)x_1 + 2x_2 + (1 + \lambda)x_5 \leq 1 & (I_3) \\
(3 - 3\lambda)x_1 + 3x_2 + 3x_3 \leq 2 - \lambda & (I_4) \\
x_1 \leq 0 & (I_5) \\
x_2 \leq 0 & (I_6) \\
x_3 \leq 0 & (I_7) \\
x_4 \leq 0 & (I_8) \\
x_5 \leq 0 & (I_9) \\
x_1 + x_2 + x_3 + x_4 + x_5 \leq 1 & (I_{10})
\end{cases}
\]

**Finding vertices:** The polytope \( M_{a,b}(\lambda) \) is a 5–dimensional polytope with 10 facets \( F_j, j = 1, 2, \ldots, 10 \) where the facet \( F_j \) is defined by the equation\(^\dagger\) associated to the inequality \((I_j)\). To determine the set of all vertices of \( M_{a,b}(\lambda) \), we consider each possible subset \( J \subseteq \{1, 2, \ldots, 10\} \) that corresponds to 5 facets \( F_j \) with \( j \in J \) and solve the corresponding set of 5 equations. If the solution is a unique point\(^\ddagger\), said \( V_j = (x_1, x_2, x_3, x_4, x_5) \), then we obtain a potential vertex; otherwise the 5 facets considered do not determine a vertex of \( M_{a,b}(\lambda) \).

Given that a potential vertex \( V_j \) is obtained, we then check whether or not \( V_j \) satisfies the remaining 5 inequalities \((E_j)\) with \( j \notin J \). Note that this amounts to solving 5 inequalities with \( \lambda \) as the variable to obtain all values of \( \lambda \) for which \( V_j \) is a vertex of \( M_{a,b}(\lambda) \).

For example, when we choose \( J = \{1, 2, 3, 4, 5\} \), we solve

\[
\begin{cases}
-2x_1 + (\lambda - 2)x_2 - (1 + \lambda)x_3 - (1 - \lambda)x_4 - \lambda x_5 = -1 & (I_1) \\
-x_1 - (1 + \lambda)x_2 + (1 - 2\lambda)x_3 + (1 - \lambda)x_4 - 2\lambda x_5 = -\lambda & (I_2) \\
(2 - \lambda)x_1 + 2x_2 + (1 + \lambda)x_5 = 1 & (I_3) \\
(3 - 3\lambda)x_1 + 3x_2 + 3x_3 = 2 - \lambda & (I_4) \\
x_1 = 0 & (I_5)
\end{cases}
\]

\(^\dagger\) Simply replace “\( \leq \)" by “\( = \)".

\(^\ddagger\) The corresponding determinant is non null.
and obtain the potential vertex $V_{(1,2,3,3,4,5)} = \left(0, \frac{1-\lambda^2}{3-\lambda}, 0, \frac{\lambda+1}{3-\lambda}, \frac{1-2\lambda}{3-\lambda}\right)$ given that $0 \leq \lambda < 1$. Now $V_{(1,2,3,3,4,5)}$ is a vertex if $V_{(1,2,3,3,4,5)}$ satisfies the 5 remaining inequalities $(I_j)$ with $j \in \{6,7,8,9,10\}$. That is when $\lambda$ simultaneously satisfies $\frac{1-\lambda^2}{3-\lambda} \leq 0$, $0 \leq 0$, $-(\frac{1}{2} + \frac{\lambda}{2}) \leq 0$, $-(\frac{1}{2} + \frac{\lambda}{2}) \leq 0$ and $\frac{1-\lambda^2}{3-\lambda} + \frac{1}{3} \lambda + \frac{1}{3} \cdot \frac{1-2\lambda}{3-\lambda} \leq 1$. This is the case for $0 \leq \lambda \leq \frac{1}{2}$. We then say that $V_{(1,2,3,3,4,5)}$ is a potential vertex and its validity domain is the real range $[0, \frac{1}{2}]$. The same operation is undertaken for each of the 210 subsets $J$ of $\{1,2,\ldots,10\}$ to obtain all vertices with their corresponding validity domains. The set of all vertices is stable for $\lambda = 0$ or $0 < \lambda < \frac{1}{2}$ or $\lambda = \frac{1}{2}$ or $\frac{1}{2} < \lambda < 1$ or $\lambda = 1$.

For illustration, we assume that $0 < \lambda < \frac{1}{2}$. In this case, we find that there are 24 distinct vertices from the 210 possible subsystems of 5 equations extracted from (48). To facilitate the triangulation process, each vertex is indexed with an integer $j \in \{1,2,\ldots,24\}$ and is associated with the set of facets it belongs to. For example, $V_{(1,2,3,3,4,5,7,10)}$ becomes $v_1 = V_{(1,2,3,3,4,5,7,10)}$ meaning that $v_1$ is the first vertex of $M_{a,b}(\lambda)$ and belongs to $F_j$ with $j \in \{1,2,3,4,5,7,10\}$. The reader can then check that the 24 vertices of $M_{a,b}(\lambda)$ are:

<table>
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<tr>
<th>$j$</th>
<th>$V_{(1,2,3,4,5,7,10)}$</th>
<th>$j$</th>
<th>$V_{(1,2,3,4,5,7,10)}$</th>
<th>$j$</th>
<th>$V_{(1,2,3,4,5,7,10)}$</th>
<th>$j$</th>
<th>$V_{(1,2,3,4,5,7,10)}$</th>
</tr>
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<td>7</td>
<td>2</td>
<td>13</td>
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<td>19</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>8</td>
<td>3</td>
<td>14</td>
<td>20</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
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<td>9</td>
<td>4</td>
<td>15</td>
<td>21</td>
<td>26</td>
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</tr>
<tr>
<td>4</td>
<td>4</td>
<td>10</td>
<td>5</td>
<td>16</td>
<td>22</td>
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<td>22</td>
</tr>
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<td>5</td>
<td>5</td>
<td>11</td>
<td>6</td>
<td>17</td>
<td>23</td>
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<td>6</td>
<td>12</td>
<td></td>
<td>18</td>
<td>24</td>
<td>29</td>
<td>24</td>
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</table>

with their coordinates given by

<table>
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<tr>
<th>$j$</th>
<th>$v_j$</th>
<th>$j$</th>
<th>$v_j$</th>
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<tbody>
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<td>1</td>
<td>$\left(0, \frac{1-\lambda^2}{3-\lambda}, 0, \frac{\lambda+1}{3-\lambda}, \frac{1-2\lambda}{3-\lambda}\right)$</td>
<td>13</td>
<td>$\left(\frac{1}{2}, 0, 0, 0, 0, 0, \frac{\lambda}{2-\lambda}\right)$</td>
</tr>
<tr>
<td>2</td>
<td>$\left(\frac{1}{2}, 0, 0, 0, \frac{1}{2}\right)$</td>
<td>14</td>
<td>$\left(\frac{1}{2}, 0, 0, 0, 0, 0, 0\right)$</td>
</tr>
<tr>
<td>3</td>
<td>$\left(0, \frac{1-\lambda^2}{3-\lambda}, \frac{\lambda+1}{3-\lambda}, 0, \frac{1-2\lambda}{3-\lambda}\right)$</td>
<td>15</td>
<td>$\left(\frac{1}{2}, 0, 0, 0, 0\right)$</td>
</tr>
<tr>
<td>4</td>
<td>$\left(\frac{1-\lambda^2}{3-\lambda}, 0, \frac{1-\lambda^2}{3-\lambda}, 0, \frac{1-2\lambda}{3-\lambda}\right)$</td>
<td>16</td>
<td>$\left(0, \frac{1}{2}, 0, 0, 0\right)$</td>
</tr>
<tr>
<td>5</td>
<td>$\left(\frac{1-\lambda^2}{3-\lambda}, 0, 0, \frac{1-\lambda^2}{3-\lambda}, 0\right)$</td>
<td>17</td>
<td>$\left(\frac{1}{2}, 0, 0, 0, 0\right)$</td>
</tr>
<tr>
<td>6</td>
<td>$\left(0, \frac{1}{2}, 0, 0, 0\right)$</td>
<td>18</td>
<td>$\left(\frac{1}{2}, 0, 0, 0, 0, 0\right)$</td>
</tr>
<tr>
<td>7</td>
<td>$\left(0, \frac{1}{2}, 0, 0, 0, 0, 0\right)$</td>
<td>19</td>
<td>$\left(\frac{1}{2}, 0, 0, 0, 0\right)$</td>
</tr>
<tr>
<td>8</td>
<td>$\left(\frac{1-\lambda^2}{3-\lambda}, 0, \frac{1-\lambda^2}{3-\lambda}, 0, 0\right)$</td>
<td>20</td>
<td>$\left(0, \frac{1-\lambda^2}{3-\lambda}, \frac{\lambda}{2-\lambda}, 0, \frac{1-2\lambda}{3-\lambda}\right)$</td>
</tr>
<tr>
<td>9</td>
<td>$\left(\frac{1}{2}, 0, \frac{1-\lambda^2}{3-\lambda}, 0, 0, 0, 0\right)$</td>
<td>21</td>
<td>$\left(\frac{1}{2}, 0, 0, 0, 0\right)$</td>
</tr>
<tr>
<td>10</td>
<td>$\left(\frac{1}{2}, 0, \frac{1-\lambda^2}{3-\lambda}, 0, 0, 0, 0\right)$</td>
<td>22</td>
<td>$\left(0, \frac{1}{2}, 0, 0, 0\right)$</td>
</tr>
<tr>
<td>11</td>
<td>$\left(0, \frac{1}{2}, 0, \frac{1-\lambda^2}{3-\lambda}, 0\right)$</td>
<td>23</td>
<td>$\left(\frac{1}{2}, 0, 0, 0, 0\right)$</td>
</tr>
<tr>
<td>12</td>
<td>$\left(0, \frac{1}{2}, \frac{1-\lambda^2}{3-\lambda}, 0, 0\right)$</td>
<td>24</td>
<td>$\left(\frac{1}{2}, 0, 0, 0, 0\right)$</td>
</tr>
</tbody>
</table>

But one must still keep in mind that a vertex is a point, $v_1$ or $V_{(1,2,3,3,4,5,7,10)}$ are both the reprentation of the same point which is $\left(0, \frac{1-\lambda^2}{3-\lambda}, 0, \frac{1}{2}, \frac{\lambda}{2}, \frac{1-2\lambda}{3-\lambda}\right)$. This is important to find the final volume.
Triangulation: In the triangulation process described in section 3.3, we choose at each step a vertex that belongs to the greatest number of facets\(^7\). At the first step, \(v_1\) is chosen as it belongs to 7 facets \(F_j\) with \(j \in \{1, 2, 3, 4, 5, 7, 10\}\). As \(v_1\) is out of 3 facets \(F_j\) with \(j \in \{6, 8, 9\}\), we obtain a dissection of \(M_{6,8}(\lambda)\) in 3 polytopes \(v_1F_6, v_1F_8\) and \(v_1F_9\). The process continues by applying the same operation to \(F_6, F_8\) and \(F_9\) which are now the current polytopes for the dissection operation.

For example, at the second step, note that \(F_6\) is a 4-dimensional polytope and its vertices are directly obtained from (50) by only considering all vertices \(V_j\) such that \(6 \in J\). Clearly \(F_6\) is the polytope defined by \(v_2, v_4, v_7, v_8, v_{10}, v_{11}, v_{15}, v_{17}, v_{19}, v_{21}, v_{23}\) and \(v_{24}\). Now each facet of \(F_6\) is obtained as any 3-dimensional polytope \(F_6 \cap F_j = \{V_j : \{6, j\} \leq J\}\) for \(j \neq 6\). Among vertices of \(v_2\), \(v_2\) belongs to 6 facets namely \(F_6 \cap F_j\) with \(j \in \{1, 2, 3, 4, 7, 10\}\). We then choose \(v_2\) as the new apex to obtain a dissection of \(F_6\) into two 3-dimensional polytopes \(v_2F_6 \cap F_8\) and \(v_2F_6 \cap F_9\).

The process continues by reiterating this operation for each new polytope to obtain a triangulation of \(M_{6,8}(\lambda)\) into 55 simplices \(S_k, k = 1, 2, ..., 55\). Note that a simplex is obtained in this process as soon as the union of the set of apexes already chosen and the current polytope contains 6 points. To ease the presentation, we give to each simplex \(S_1\) in the triangulation process three components as \(S_1 = [S_{1,1}, S_{1,2}, S_{1,3}]\) where \(S_{1,1}\) is the list of apexes chosen, \(S_{1,2}\) is the list of facets \(F_j\) encountered and \(S_{1,3}\) is the list of vertices in the current polytope that define \(S_1\). Clearly, the set of vertices of \(S_1\) is obtained by merging \(S_{1,1}\) and \(S_{1,3}\). For example the first simplex we obtain is \(S_1 = [[1, 2, 4, 10], [6, 8, 3, 7], [13, 23]]\)\(^8\) meaning that \(S_1\) is obtained in four steps: (step1) \(v_1\) is the chosen apex and \(F_6\) is the 4-dimensional current polytope; (step2) \(v_2\) is the apex chosen and \(F_6 \cap F_8\) is the 3-dimensional current polytope; (step3) \(v_4\) is the apex chosen and \(F_6 \cap F_8 \cap F_3\) is the 2-dimensional current polytope; (step4) \(v_{10}\) is the apex chosen and \(F_6 \cap F_8 \cap F_3 \cap F_1\) is the 1-dimensional current polytope, the line segment \(v_{13}v_{23}\). Thus \(S_1\) is the simplex generated by \(v_1, v_2, v_4, v_{10}, v_{13}\) and \(v_{23}\). That is \(S_1 = \Delta(v_1, v_2, v_4, v_{10}, v_{13}, v_{23})\) with its corresponding volume

\[
\text{vol}(S_1) = \frac{\lambda^2 (1 - \lambda + \lambda^2)^2}{9720 (1 - \lambda)(\lambda + 1)(2 - \lambda)(\lambda^2 + 2)}.
\]

All the 55 simplices obtained are listed below:

\[
[[1, 2, 4, 10], [6, 8, 3, 7], [13, 23]], [[1, 2, 4, 10], [6, 8, 3, 9], [23, 24]], [[1, 2, 4, 10], [6, 8, 3, 10], [21, 24]], [[1, 2, 4, 10], [6, 8, 7], [13, 15, 23]], [[1, 2, 4, 8], [6, 8, 9, 3], [23, 24]], [[1, 2, 4, 8], [6, 8, 9, 7], [19, 24]], [[1, 2, 17, 8], [6, 9, 8, 3], [23, 24]], [[1, 2, 17, 8], [6, 9, 8, 7], [15, 23]], [[1, 2, 17, 8], [6, 9, 8, 10], [19, 24]], [[1, 3, 9, 10], [8, 3, 6, 7], [13, 23]], [[1, 3, 9, 10], [8, 3, 6, 9], [23, 24]], [[1, 3, 9, 10], [8, 3, 6, 10], [21, 24]], [[1, 3, 9, 10], [8, 3, 7], [13, 14, 23]], [[1, 3, 9, 12], [8, 3, 9, 6], [23, 24]], [[1, 3, 9, 12], [8, 3, 9, 7], [14, 23]], [[1, 3, 9, 12], [8, 3, 9, 10], [22, 24]], [[1, 3, 9, 20], [8, 3, 10, 6], [21, 24]], [[1, 3, 9, 20], [8, 3, 10, 9], [22, 24]], [[1, 3, 9, 10], [8, 6, 3, 7], [13, 23]], [[1, 3, 4, 10], [8, 6, 3, 9], [23, 24]], [[1, 3, 4, 10], [8, 6, 3, 10], [21, 24]], [[1, 3, 4, 8], [8, 6, 9, 7], [15, 23]], [[1, 3, 4, 8], [8, 6, 9, 10], [19, 24]], [[1, 3, 8, 7], [13, 14, 15, 23]], [[1, 3, 6, 12], [8, 9, 3, 6], [23, 24]], [[1, 3, 6, 12], [8, 9, 3, 7], [14, 23]], [[1, 3, 6, 12], [8, 9, 3, 10], [22, 24]], [[1, 3, 6, 8], [8, 9, 6, 3], [23, 24]], [[1, 3, 6, 8], [8, 9, 6, 7], [15, 23]], [[1, 3, 6, 8], [8, 9, 6, 10], [19, 20]].

\(^7\) Note that the choice of the apex does not change the corresponding volume even though it may influence the total number of simplices in the triangulation.

\(^8\) Or equivalently \(S_1 = [[v_1, v_2, v_4, v_{10}], [F_6, F_8, F_9], [v_{13}, v_{23}]]\).
The volume of $M_{a,b}(\lambda)$ is then obtained by applying (28) to each of these simplices using vertices presented at (50) with their corresponding coordinates given by (51).

We proceed in the same way to compute the volume of each polytope considered in this paper\(^9\). For example, when $\frac{1}{2} < \lambda < 1$, $M_{a,b}(\lambda)$ has 25 vertices and we obtain a triangulation into 56 simplices to derive the corresponding volume. In the same way, it is worth noticing that the set $\mathcal{S}$ of all voting situations: (i) is defined by $(I_j)$ with $j \in \{5,6,7,8,9,10\}$ (for each $(I_j)$, see (48)); (ii) is the unit-simplex $\Delta(w_0,w_1,w_2,w_3,w_5)$ where $w_0 = (0,0,0,0,0), w_1 = (1,0,0,0,0), w_2 = (0,1,0,0,0), w_3 = (0,0,1,0,0), w_4 = (0,0,0,1,0) \text{ and } w_5 = (0,0,0,0,1)$; and (iii) has, by applying (28), the volume $\text{vol}(\mathcal{S}) = \frac{1}{120}$.

**Appendix C: Limited size of manipulating coalitions with usual PVRs**

As announced in Section 3.5, for $\lambda \in \left\{ \frac{m}{12}, k = 0, 1, 2, \ldots, 10 \right\}$, we have (i) determined necessary and sufficient conditions for $x \in M^P_{a,b}(\lambda)$ and then deduce equivalent conditions for $M^P_{a,c}(\lambda)$ and $M^P_{d,c}(\lambda)$; (ii) computed $\text{vol}(M^P_{a,b}(\lambda))$ and $\text{vol}(M^P_{a,b}(\lambda) \cap M^P_{c,c}(\lambda))$; and then (iii) applied (39) and (41) to derive $\text{vol}(p, F_0, IAC)$. For usual PVRs, namely the plurality rule $F_0$, the Borda rule $F_0$, and the antiplurality rule $F_1$, the results are the followings:

Consider $p \in [0, 1]$. Then

$$
\text{vol}(p, F_0, IAC) = \begin{cases} 
\frac{-25}{18} p^5 + \frac{25}{6} p^4 - \frac{40}{9} p^3 - \frac{5}{6} p^2 + \frac{25}{12} p & \text{if } 0 \leq p \leq \frac{1}{4} \\
\frac{1}{9} + \frac{25}{18} p - \frac{5}{9} p^2 + \frac{40}{9} p^3 - \frac{10}{9} p^4 - \frac{8}{9} p^5 & \text{if } \frac{1}{4} \leq p \leq \frac{1}{2} \\
\frac{7}{27} & \text{if } \frac{1}{2} \leq p \leq 1
\end{cases}
$$

\(^9\) We have built some routines using MAPLE codes to undertake all these operations. All those routines are available from the author upon simple request.
Appendix D: Proof of Proposition 7

**Proof** Assume that $1 > \lambda > \frac{1}{7}$. We use the same set of notations as in Subsection 3.4 (see page 5, paragraph 2) to prove that the maximum minimum size of a manipulating coalition is at most $\frac{1}{3(\lambda^2 - \lambda)}$.

Now from the list of constraints that depend on $y_1$, the lower bound of $y_1$ is $y_1^\text{lb} = \max(0, p - x_6, r)$ and its upper bound is $y_1^\text{ub} = \min(p, x_3, t)$ where $r$ and $t$ are given by (31). We deduce that $t > r$, $t > p - x_6$, $t > 0$, $x_3 > r$, $x_3 \geq p - x_6$ and $p > r$. There for the lower bound of $p$ is $p^\text{lb} = \min(v_1(x), v_2(x), v_3(x), v_4(x), 0)$ where

- $v_1(x) = \frac{(\lambda^2 + 2\lambda - 2)x_1 + (1 - \lambda + \lambda^2)(x_3 + x_4 - 2x_5) + (1 + 2\lambda - 2\lambda^2)x_4 + (1 - 4\lambda + \lambda^2)x_5}{3(\lambda-1)}$
- $v_2(x) = \frac{(\lambda - 1)x_1 - x_2 + x_3 + x_4 - x_5 + \lambda x_6}{3(\lambda-1)}$
- $v_3(x) = \frac{(1 - 2\lambda)x_1 + x_2 + (1 - 2\lambda)x_3 - x_4 + \lambda x_6}{2(\lambda - 1)}$
- $v_4(x) = \frac{\lambda x_1 - 2x_2 + x_3 + (1 - \lambda)x_4 + x_5 + (\lambda - 1)x_6}{2(\lambda - 1)}$

Therefore

$$p^\text{lb}(\lambda) = \max(v_1^\text{*}(x), v_2^\text{*}(x), v_3^\text{*}(x), v_4^\text{*}(x), 0)$$

where $v_j^\text{*}(x)$ is the solution to the following linear program

$$\max v_j(x)$$

s.t. $x \in M_{a,b}(F_{\lambda})$

Solving (35) for $x_1, x_2, x_3, x_6,$ and $x_2$ leads to

$$v_4(x) = \frac{2 - \lambda}{3} \frac{3(1 - \lambda)x_4 + 3\lambda(1 - \lambda)x_5 + (2 - \lambda)x_1 + (3\lambda(1 - \lambda)x_4 + 3(1 - \lambda)x_5 + 3(1 - \lambda)(2 - \lambda)x_1)}{2(1 + \lambda)(1 - \lambda)}$$
Similarly, by solving constraints above for \( x_2, x_5, x_6 \) and \( s_2 \) yields

\[
\begin{align*}
\nu_1(x) &= \frac{2-s_2}{s} x_4 - (1 - s_2) x_3 - \frac{2(1-s_2)^2}{s(1-x_3)} - \frac{2-s_2}{s} x_4 \\
\nu_2(x) &= \frac{2-s_2}{s} x_4 - (1 - s_2) x_3 - \frac{2(1-s_2)^2}{s(1-x_3)} - \frac{2-s_2}{s} x_4 \\
\nu_3(x) &= \frac{2-s_2}{s} x_4 - (1 - s_2) x_3 - \frac{2(1-s_2)^2}{s(1-x_3)} - \frac{2-s_2}{s} x_4
\end{align*}
\]

We then deduce that for all \( x \in M_{d,b}(F_3) \), \( \nu_j(x) < \frac{2-s_2}{s} \), \( \forall j \in \{1, 2, 3, 4\} \).

References